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A NOTE ON THE PAPER ENTITLED SIXTEENTH-ORDER METHOD FOR NONLINEAR EQUATIONS

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Abstract. The purpose of this paper is to provide some corrections regarding algebraic flaws encountered in the paper entitled "Sixteenth-order method for nonlinear equations" which was published in January of 2010 by Li et al.[9]. Further detailed comments on their error equation are stated together with convergence analysis as well as high-precision numerical experiments.

1. Introduction

Iterative methods including Newton's method, Jarratt's fourth-order method[3] and King's fourth-order method[4] have been developed and successfully applied to find a root of a given nonlinear equation. A variety of other high-order iterative methods have been investigated by many researchers such as Chun[1], Geum[2], Li et al.[9], Ren[6] and Sharma[7].

Suppose that $f : \mathbb{R} \to \mathbb{R}$ has a simple root α and is sufficiently smooth in an open interval $\mathbf{D} \subseteq \mathbb{R}$ containing α. Li et al.[9] has recently suggested a three-step sixteenth-order method shown below: for $n = 0, 1, \cdots$,

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{2f(x_n)-f(y_n)}{2f(x_n)-5f(y_n)} \cdot \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = w_n - \frac{2f(z_n)-f(w_n)}{2f(z_n)-5f(w_n)} \cdot \frac{f(w_n)}{f'(z_n)}, \end{cases} \tag{1.1}$$

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where $w_n = z_n - \frac{f(z_n)}{f'(z_n)}$ and $x_0 \in \mathbf{D}$ is given close to α.

Although their main theorem regarding the error equation of (1.1) appears to be correct, its proof involves critical algebraic flaws in the coefficients of $e_n^5, e_n^9, e_n^{10}, e_n^{13}, e_n^{14}, e_n^{15}$ associated with the expressions $z_n, f(z_n)$, $f'(z_n)$ and $f(w_n)$. In addition, the coefficient of e_n^4 in the expression $\frac{2f(x_n)-f(y_n)}{2f(x_n)-5f(y_n)}$ should be corrected as $(15c_2^4)/4-11c_2^2c_3+4c_3^2+2c_2c_4+8c_5$, with $c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}$ for $j = 2, 3, \cdots$.

Despite the presence of these faulty coefficients, the dominating term in their error equation happens to be luckily correct, since it is dependent only on the coefficients of e_n^4, e_n^8, e_n^{12} in $z_n, f(z_n), f'(z_n)$ and $f(w_n)$. The details of these flaws are described in Section 2. The main aim of this paper is to provide corrections regarding some algebraic flaws encountered in the paper by Li et al.[9]. To convince the analysis presented here, some results of high-precision numerical experiments are displayed for several test functions chosen from the paper in [9].

2. Convergence analysis

The convergence property of iterative method (1.1) is best illustrated in Theorem 2.1 stated below:

THEOREM 2.1. *Let $f : \mathbf{D} \subseteq \mathbb{R} \to \mathbb{R}$ with an open interval $\mathbf{D}$ be a sufficiently smooth function having a real zero $\alpha \in \mathbf{D}$. Let $c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}$ for $j = 2, 3, \cdots$. Assume that c_2, c_3 and c_4 are not vanishing. Let x_0 be an initial guess chosen in a sufficiently small neighborhood of α. Then iteration scheme (1.1) is of sixteenth-order and its error equation satisfies the following:*

$$\epsilon_{n+1} = Ae_n^{16} + \kappa e_n^{17} + O(e_n^{18}), \tag{2.1}$$

where $A = -(c_2c_3)^5$, $\kappa = 4(c_2c_3)^4\phi$ and $\phi = (3c_2^4)/2+2c_2^2c_3-2c_3^2-2c_2c_4$.

Proof. Taylor series expansion of $f(x_n)$ about α up to fifth-order terms yields with $f(\alpha) = 0$:

$$f(x_n) = f'(\alpha)(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + O(e_n^6)), \tag{2.2}$$

where $e_n = x_n - \alpha$ for $n = 0, 1, 2, \cdots$. For ease of notation, e_n will be denoted by e (not to be confused with Napier's base for natural logarithms) for the time being. With the aid of symbolic computation of *Mathematica*[10], a lengthy algebraic computation induces relations (2.3)-(2.8) below:

$$f'(x_n) = f'(\alpha)(1 + 2c_2e + 3c_3e^2 + 4c_4e^3 + 5c_5e^4 + O(e^5)), \qquad (2.3)$$

$$\frac{f(x_n)}{f'(x_n)} = e - c_2e^2 + 2\lambda e^3 + A_1e^4 + A_2e^5 + O(e^6), \qquad (2.4)$$

where $\lambda = c_2^2 - c_3, A_1 = -4c_2^3 + 7c_2c_3 - 3c4, A_2 = 8c_2^4 - 20c_2^2c_3 + 6c_3^2 + 10c_2c_4 - 4c_5$.

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2e^2 - 2\lambda e^3 - A_1e^4 - A_2e^5 + O(e^6), \qquad (2.5)$$

$$f(y_n) = f'(\alpha)\big(c_2^2e^2 - 2\lambda e^3 + (c_2^3 - A_1)e^4 + (-A_2 - 4\lambda c_2^2)e^5 + O(e^6)\big). \qquad (2.6)$$

$$\begin{aligned}\frac{f(y_n)}{f'(x_n)} &= c_2e^2 + (-4c_2^2 + 2c_3)e^3 \\ &\quad + (13c_2^3 - 14c_2c_3 + 3c_4)e^4 + B_5e^5 + O(e^6),\end{aligned} \qquad (2.7)$$

where $B_5 = -38c_2^4 + 64c_2^2c_3 - 12c_3^2 - 20c_2c_4 + 4c_5$.

$$\begin{aligned}\frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)} &= 1 + 2c_2e + (-c_2^2 + 4c_3)e^2 \\ &\quad + (-3c_2^3/2 + 6c_4)e^3 + B_4e^4 + O(e^5),\end{aligned} \qquad (2.8)$$

where $B_4 = (15c_2^4)/4 - 11c_2^2c_3 + 4c_3^2 + 2c_2c_4 + 8c_5$.

Consequently, we have after substituting λ, A_1, A_2, B_4 and B_5:

$$z_n = y_n - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)} \cdot \frac{f(y_n)}{f'(x_n)} = \alpha - c_2c_3e^4 + \phi e^5 + O(e^6), \qquad (2.9)$$

where $\phi = 3c_2^4/2 + 2c_2^2c_3 - 2c_3^2 - 2c_2c_4$.

Letting $\tau = z_n - \alpha$ $(n = 0, 1, 2, \cdots)$ and Taylor series expansion of $f(x_n)$ about α yields with e_n replaced by τ in (2.2) and(2.3):

$$f(z_n) = f'(\alpha)(\tau + c_2\tau^2 + c_3\tau^3 + c_4\tau^4 + c_5\tau^5 + O(\tau^6)), \qquad (2.10)$$

$$f'(z_n) = f'(\alpha)(1 + 2c_2\tau + 3c_3\tau^2 + 4c_4\tau^3 + 5c_5\tau^4 + O(\tau^5)), \qquad (2.11)$$

$$\frac{f(z_n)}{f'(z_n)} = \tau - c_2\tau^2 + 2\lambda\tau^3 + A_1\tau^4 + A_2\tau^5 + O(\tau^6), \qquad (2.12)$$

$$w_n = z_n - \frac{f(z_n)}{f'(z_n)} = \alpha + c_2\tau^2 - 2\lambda\tau^3 - A_1\tau^4 - A_2\tau^5 + O(\tau^6). \qquad (2.13)$$

Similarly, replacing e_n by $w_n - \alpha$ in (2.2) and(2.3)yields:

$$\begin{aligned} f(w_n) = f'(\alpha)(c_2\tau^2 - 2\lambda\tau^3 + (-A_1 + c_2^3)\tau^4 \\ + (-A_2 - 4c_2^2\lambda)\tau^5 + O(\tau^6)), \end{aligned} \tag{2.14}$$

$$\begin{aligned} \frac{f(w_n)}{f'(z_n)} &= c_2\tau^2 + (-4c_2^2 + 2c_3)\tau^3 \\ &+ (13c_2^3 - 14c_2c_3 + 3c_4)\tau^4 + B_5\tau^5 + O(\tau^6), \end{aligned} \tag{2.15}$$

$$\begin{aligned} \frac{2f(z_n) - f(w_n)}{2f(z_n) - 5f(w_n)} &= 1 + 2c_2\tau + (-c_2^2 + 4c_3)\tau^2 \\ &+ (-3c_2^3/2 + 6c_4)\tau^3 + B_4\tau^4 + O(\tau^5). \end{aligned} \tag{2.16}$$

Hence the third equation of (1.1) leads us to the relation below:

$$\begin{aligned} e_{n+1} = x_{n+1} - \alpha = w_n - \alpha - \frac{2f(z_n) - f(w_n)}{2f(z_n) - 5f(w_n)} \cdot \frac{f(w_n)}{f'(z_n)} \\ = -c_2c_3\tau^4 + \phi\tau^5 + O(\tau^6). \end{aligned} \tag{2.17}$$

Substituting $\tau = z_n - \alpha = -c_2c_3e^4 + \phi e^5 + O(e^6)$ into (2.17) yields the desired relation below with e denoted by e_n:

$$\epsilon_{n+1} = Ae_n^{16} + \kappa e_n^{17} + O(e_n^{18}), \tag{2.18}$$

where $A = -(c_2c_3)^5$, $\kappa = 4(c_2c_3)^4\phi$ and $\phi = (3c_2^4)/2 + 2c_2^2c_3 - 2c_3^2 - 2c_2c_4$, from which the proof is completed. □

It is worth to observe that

$$A = \lim_{n\to\infty} \frac{e_n}{{e_{n-1}}^{16}}$$

and

$$\kappa = \lim_{n\to\infty} \frac{e_n - A{e_{n-1}}^{16}}{{e_{n-1}}^{17}}.$$

Note that $\eta = |A|$ is known as the *asymptotic error constant*. As a result of the above theorem, some faulty coefficients of $e_n^5, e_n^9, e_n^{10}, e_n^{13}, e_n^{14}, e_n^{15}$ have been found in equations (15)-(18) of the paper by [9]. Those faulty coefficients should be corrected as shown in Table 1. The following remarks deserve special attention.

Remark 2.2. **(1)** If the coefficient ϕ of e_n^5 in (2.9) were chosen as $\rho = -73c_2^4/2 + 66c_2^2c_3 - 14c_3^2 - 22c_2c_4$ that Li et al. used, then the coefficient of e_n^{17} in (2.18) would be $4(c_2c_3)^4\rho$.
(2) The coefficient of $A\ e_n^{16}$ in (2.18) remains unchanged as Li et al. suggested, for it depends only on the coefficients of e_n^4, e_n^8, e_n^{12} in the expressions $z_n, f(z_n), f'(z_n)$ and $f(w_n)$.

TABLE 1. Corrected coefficients of e_n^k for $k = 5, 9, 10, 13, 14, 15$

Expression	e_n^5	e_n^9	e_n^{10}	e_n^{13}	e_n^{14}	e_n^{15}
z_n	ϕ	N/A	N/A	N/A	N/A	N/A
$f(z_n)$	ϕ	$-2c_2^2c_3\phi$	$c_2\phi^2$	$3c_2^2c_3^3\phi$	$-3c_2c_3^2\phi^2$	$c_3\phi^3$
$f'(z_n)$	$2c_2\phi$	$-6c_2c_3^2\phi$	$3c_3\phi^2$	$12c_2^2c_3^2c_4\phi$	$-12c_2c_3c_4\phi^2$	$4c_4\phi^3$
$f(w_n)$	N/A	$-2c_2^2c_3\phi$	$c_2\phi^2$	$-6c_2^2c_3^2\lambda\phi$	$6c_2c_3\lambda\phi^2$	$-2\lambda\phi^3$

Here $\lambda = c_2^2 - c_3,\ \phi = 3c_2^4/2 + 2c_2^2c_3 - 2c_3^2 - 2c_2c_4$ and N/A = not applicable.

3. Algorithm, numerical results and discussions

The analysis described in Section 2 allows us to develop a zero-finding algorithm to be implemented with *Mathematica*:

Algorithm 3.1 (Zero-Finding Algorithm)
Step 1. Construct iteration scheme (1.1) with the given function f having a simple zero α for $n \in \mathbb{N} \cup \{0\}$ as mentioned in Section 1.
Step 2. Set the minimum number of precision digits. With exact or most accurate zero α, supply the theoretical asymptotic error constant $\eta = |A|$, order of convergence p as well as c_2, c_3, c_4 and c_5 stated in Section 2. Set the error bound ϵ, the maximum iteration number n_{max} and the initial guess x_0. Compute $f(x_0)$ and $x_0 - \alpha$.
Step 3. Tabulate the computed values of $n,\ x_n,\ e_n = x_n - \alpha,\ \frac{e_n}{e_{n-1}{}^p},\ A,$ $\frac{e_n - Ae_{n-1}{}^p}{e_{n-1}{}^{p+1}}$ and κ.

Throughout the numerical experiments, the minimum number of precision digits was properly chosen by specifying Mathematica command *\$MinPrecision* within the range of 800 and 1500, being large enough to minimize round-off errors as well as to clearly observe the computed asymptotic error constants requiring small-number divisions. A constant error bound $\epsilon = 0.5 \times 10^{-300}$ was used in the current experiments for relatively accurate computation to verify the convergence order of 16. The values of initial guess x_0 were selected close to α in order to guarantee convergence. The computed asymptotic error constant or coefficients of the error equation (2.1) appear to agree up to 10 significant digits with the theoretical ones. The computed zeros are actually rounded to be accurate up to the 300 significant digits, although displayed only up to 15 significant digits. In case that the exact value of α is not available, it is computed with Mathematica command *FindRoot* to have accuracy of

700 significant digits, while being listed up to 15 significant digits due to the limited paper space.

Three test functions $f(x) = (x-1)^3 - 1$, $f(x) = \sqrt{x^2+2x+5} - \sin x - x^2 + 3$ and $f(x) = \ln x + \sqrt{x} - 5$ are selected from the paper[8] for numerical experiments to be implemented with iteration scheme (1.1). As expected, the computational results clearly show coefficients A, κ and asymptotic error constants $\eta = |A|$ with sixteenth-order convergence. Tables 2, 3 and 4 list iteration indexes n, approximate zeros x_n, errors $e_n = x_n - \alpha$ and $\frac{e_n}{{e_{n-1}}^{16}}$ as well as A plus $\frac{e_n - A{e_{n-1}}^{16}}{{e_{n-1}}^{17}}$ and κ.

TABLE 2. Convergence for $f(x) = (x-1)^3 - 1$ with $\alpha = 2$

n	x_n	$e_n = x_n - \alpha$	$\frac{e_n}{{e_{n-1}}^{16}}$	A	$\frac{e_n - A{e_{n-1}}^{16}}{{e_{n-1}}^{17}}$	κ
0	1.8	0.2		-0.00411522633		0.0960219478
1	1.99999999999700	-2.99×10^{-12}	-0.4576767021		2.267807379	
2	2.00000000000000	-1.76×10^{-187}	-0.00411522633		0.0960219478	
3	2.00000000000000	$0.\times10^{-749}$				

TABLE 3. Convergence for $f(x) = \sqrt{x^2+2x+5} - \sin x - x^2 + 3$ with $\alpha \approx 2.33196765588396$

n	x_n	$e_n = x_n - \alpha$	$\frac{e_n}{{e_{n-1}}^{16}}$	A	$\frac{e_n - A{e_{n-1}}^{16}}{{e_{n-1}}^{17}}$	κ
0	1.8	0.531968		$-8.814878861\times10^{-11}$		$-7.912879308\times10^{-10}$
1	2.33196765588396	-1.35×10^{-18}	$-3.302562913\times10^{-14}$		$-1.656412040\times10^{-10}$	
2	2.33196765588396	-1.18×10^{-296}	$-8.814878861\times10^{-11}$		$-7.912879308\times10^{-10}$	
3	2.33196765588396	-6.98×10^{-1013}				

TABLE 4. Convergence for $f(x) = \ln x + \sqrt{x} - 5$ with $\alpha \approx 8.30943269423157$

n	x_n	$e_n = x_n - \alpha$	$\frac{e_n}{{e_{n-1}}^{16}}$	A	$\frac{e_n - A{e_{n-1}}^{16}}{{e_{n-1}}^{17}}$	κ
0	7.0	1.30943		$3.599254246\times10^{-20}$		$-2.753290168\times10^{-20}$
1	8.30943269423157	8.14×10^{-18}	$1.090749998\times10^{-19}$		$-5.581230532\times10^{-10}$	
2	8.30943269423157	1.35×10^{-293}	$3.599254246\times10^{-20}$		$-2.753290168\times10^{-20}$	
3	8.30943269423157	-1.61×10^{-1011}				

A careful inspection of Tables 2-4 clearly shows that theoretical values of κ have been observed. Hence the error equation defined by (2.1) well reflects the validity of the current analysis. In a sense of confirming high-order convergence, it is more appealing than that of the paper by

Li et al.[9] who used only 17 digits of precision. Li's numerical experiments seem to require a large number of precision digits for verifying 16th-order convergence. Besides that, a software program such as Maple or Mathematica capable of doing arbitrary-precision arithmetic is much more appropriate in confirming high-order convergence rather than Visual C++ 6.0 dealing with at most long double precision (equivalent to approximately 17 decimal digits).

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